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AUTHOR(S):

Yagasaki, Tatsuhiko

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HOMOTOPY TYPES OF DIFFEOMORPHISM GROUPS OF NONCOMPACT 2-MANIFOLDS

矢ヶ崎 達彦 (TATSUHIKO YAGASAKI)

京都工芸繊維大学 工学部

1. INTRODUCTION

This is a report on the study of topological properties of the diffeomorphism groups of noncompact smooth 2-manifolds endowed with the compact-open C^∞ -topology [18].

When M is a compact smooth 2-manifold, the diffeomorphism group $\mathcal{D}(M)$ with the compact-open C^∞ -topology is a smooth Fréchet manifold [6, Section I.4], and the homotopy type of the identity component $\mathcal{D}(M)_0$ has been classified by S. Smale [15], C. J. Earle and J. Eell [4], et. al. In the C^0 -category, for any compact 2-manifold M , the homeomorphism group $\mathcal{H}(M)$ with the compact-open topology is a topological Fréchet manifold [3, 11, 19], and the homotopy type of the identity component $\mathcal{H}(M)_0$ has been classified by M. E. Hamstrom [7].

Recently we have shown that $\mathcal{H}(M)_0$ is a topological Fréchet-manifold even if M is a noncompact connected 2-manifold, and have classified its homotopy type [17]. The argument in [17] is based on the following ingredients: (i) the ANR-property and the contractibility of $\mathcal{H}(M)_0$ for compact M , (ii) the bundle theorem connecting the homeomorphism group $\mathcal{H}(M)_0$ and the embedding spaces of submanifolds into M [16, Corollary 1.1], and (iii) a result on the relative isotopies of 2-manifolds [17, Theorem 3.1]. The same strategy based on the C^∞ -versions of these results implies a corresponding conclusion for the diffeomorphism groups of noncompact smooth 2-manifolds.

Suppose M is a smooth 2-manifold and X is a closed subset of M . We denote by $\mathcal{D}_X(M)$ the group of C^∞ -diffeomorphisms h of M onto itself with $h|_X = id_X$, endowed with the compact-open C^∞ -topology [9, Ch.2, Section 1], and by $\mathcal{D}_X(M)_0$ the identity connected component of $\mathcal{D}_X(M)$.

The following is our main result:

Theorem 1.1. *Suppose M is a noncompact connected smooth 2-manifold without boundary.*

(1) $\mathcal{D}(M)_0$ is a topological ℓ_2 -manifold.

(2) (i) $\mathcal{D}(M)_0 \simeq \mathbb{S}^1$ if $M =$ a plane, an open Möbius band or an open annulus.

(ii) $\mathcal{D}(M)_0 \simeq *$ in all other cases.

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Any separable infinite-dimensional Fréchet space is homeomorphic to the Hilbert space $\ell_2 \equiv \{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty\}$. A topological ℓ_2 -manifold is a separable metrizable space which is locally homeomorphic to ℓ_2 . Topological types of ℓ_2 -manifolds are classified by their homotopy types. Theorem 1.1 implies the following conclusion:

Corollary 1.1. (i) $\mathcal{D}(M)_0 \cong \mathbb{S}^1 \times \ell_2$ if $M =$ a plane, an open Möbius band or an open annulus.
(ii) $\mathcal{D}(M)_0 \cong \ell_2$ in all other cases.

For the subgroup of diffeomorphisms with compact supports, we have the following results: Let $\mathcal{D}(M)_0^c$ denote the subgroup of $\mathcal{D}(M)_0$ consisting of $h \in \mathcal{D}(M)$ which admits a C^∞ -isotopy with a compact support, $h_t : M \rightarrow M$ such that $h_0 = id_M$ and $h_1 = h$.

We say that a subspace A of a space X has the homotopy negligible (h.n.) complement in X if there exists a homotopy $\varphi_t : X \rightarrow X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ ($0 < t \leq 1$). In this case, the inclusion $A \subset X$ is a homotopy equivalence, and X is an ANR iff A is an ANR.

Theorem 1.2. Suppose M is a noncompact connected smooth 2-manifold without boundary. Then $\mathcal{D}(M)_0^c$ has the h.n. complement in $\mathcal{D}(M)_0$.

Corollary 1.2. (1) $\mathcal{D}(M)_0^c$ is an ANR.

(2) The inclusion $\mathcal{D}(M)_0^c \subset \mathcal{D}(M)_0$ is a homotopy equivalence.

Section 2 contains fundamental facts on diffeomorphism groups of 2-manifolds and ℓ_2 -manifolds. Section 3 contains a sketch of proofs of Theorems 1.1 and 1.2.

2. FUNDAMENTAL PROPERTIES OF DIFFEOMORPHISM GROUPS

In this preliminary section we list fundamental facts on diffeomorphism groups of 2-manifolds (general properties, bundle theorem, homotopy type, relative isotopies, etc) and basic facts on ANR's and ℓ_2 -manifolds. Throughout the paper all spaces are separable and metrizable and maps are continuous.

2.1. General property of diffeomorphism groups.

Suppose M is a smooth n -manifold possibly with boundary and X is a closed subset of M .

Lemma 2.1. (c.f. [9, Ch 2., Section 4], etc)

$\mathcal{D}_X(M)$ is a topological group, which is separable, completely metrizable, infinite-dimensional and not locally compact.

When N is a smooth submanifold of M , the symbol $\mathcal{E}_X(N, M)$ denotes the space of C^∞ -embeddings $f : N \hookrightarrow M$ with $f|_X = \text{id}_X$ with the compact-open C^∞ -topology, and $\mathcal{E}_X(N, M)_0$ denotes the connected component of the inclusion $i_N : N \subset M$ in $\mathcal{E}_X(N, M)$.

Lemma 2.2. (i) Suppose M is a smooth manifold without boundary, N is a compact smooth submanifold of M and X is a closed subset of N . Then $\mathcal{E}_X(N, M)$ is a Fréchet manifold.
(ii) Suppose M is a compact smooth n -manifold and X is a closed subset of M with $\partial M \subset X$ or $\partial M \cap X = \emptyset$. Then $\mathcal{D}_X(M)$ is a Fréchet manifold.

In Lemma 2.2 $\mathcal{E}_X(N, M)_0$ and $\mathcal{D}_X(M)_0$ are path-connected. Thus any $h \in \mathcal{D}_X(M)_0$ can be joined with id_M by a path h_t ($t \in [0, 1]$) in $\mathcal{D}_X(M)_0$.

2.2. Bundle theorems.

The bundle theorem asserts that the natural restriction maps from diffeomorphism groups to embedding spaces are principal bundles [2, 12]. This has been used to study the homotopy types of diffeomorphism groups. This theorem also plays an essential role in our argument.

Suppose M is a smooth m -manifold without boundary, N is a compact smooth n -submanifold of M and X is a closed subset of N .

Case 1: $n < m$ [2, 12]

Let U be any open neighborhood of N in M .

Theorem 2.1. For any $f \in \mathcal{E}_X(N, U)$ there exist a neighborhood \mathcal{U} of f in $\mathcal{E}_X(N, U)$ and a map $\varphi : \mathcal{U} \rightarrow \mathcal{D}_{X \cup (M \setminus U)}(M)_0$ such that $\varphi(g)f = g$ ($g \in \mathcal{U}$) and $\varphi(f) = \text{id}_M$.

Corollary 2.1. The restriction map $\pi : \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \rightarrow \mathcal{E}_X(N, U)_0$, $\pi(h) = h|_N$, is a principal bundle with fiber $\mathcal{D}_{X \cup (M \setminus U)}(M)_0 \cap \mathcal{D}_N(M)$.

Case 2: $n = m$

In this case we have a weaker conclusion: Suppose N' is a compact smooth n -submanifold of M obtained from N by attaching a closed collar $\partial N \times [0, 1]$ to ∂N . Let U be any open neighborhood of N' in M . We can apply Theorem 2.1 to $\partial N'$ to obtain the following result:

Theorem 2.2. For any $f \in \mathcal{E}_X(N', U)$ there exist a neighborhood \mathcal{U}' of f in $\mathcal{E}_X(N', U)$ and a map $\varphi : \mathcal{U}' \rightarrow \mathcal{D}_{X \cup (M \setminus U)}(M)_0$ such that $\varphi(g)f|_N = g|_N$ ($g \in \mathcal{U}'$) and $\varphi(f) = \text{id}_M$.

For the sake of simplicity, we set $\mathcal{D}_0 = \mathcal{D}_{X \cup (M \setminus U)}(M)_0$, $\mathcal{E}_0 = \mathcal{E}_X(N, U)_0$, $\mathcal{E}'_0 = \mathcal{E}_X(N', U)_0$.

Consider the restriction map $p : \mathcal{E}'_0 \rightarrow \mathcal{E}_0$, $p(f) = f|_N$ and $\pi : \mathcal{D}_0 \rightarrow \mathcal{E}_0$, $\pi(h) = h|_N$. We have the pullback diagram:

$$\begin{array}{ccc}
p^*(\mathcal{D}_0) & \xrightarrow{p_*} & \mathcal{D}_0 \\
\pi_* \downarrow & & \downarrow \pi \\
\mathcal{E}'_0 & \xrightarrow{p} & \mathcal{E}_0,
\end{array}$$

where $p^*\mathcal{D}_0 = \{(f, h) \in \mathcal{E}'_0 \times \mathcal{D}_0 \mid f|_N = h|_N\}$, $p_*(f, h) = h$ and $\pi_*(f, h) = f$. The map p_* admits a natural right inverse $q : \mathcal{D}_0 \rightarrow p^*\mathcal{D}_0$, $q(h) = (h|_{N'}, h)$. The group $\mathcal{D}_0 \cap \mathcal{D}_N(M)$ acts on $p^*\mathcal{D}_0$ by $(f, h)g = (f, hg)$ ($g \in \mathcal{D}_0 \cap \mathcal{D}_N(M)$).

Corollary 2.2.

- (1) $\pi_* : p^*(\mathcal{D}_0) \rightarrow \mathcal{E}'_0$ is a principal bundle with fiber $\mathcal{D}_0 \cap \mathcal{D}_N(M)$.
- (2) $p_* : p^*(\mathcal{D}_0) \rightarrow \mathcal{D}_0$ is a homotopy equivalence with the homotopy inverse $q : \mathcal{D}_0 \rightarrow p^*(\mathcal{D}_0)$.
- (3) $p : \mathcal{E}'_0 \rightarrow \mathcal{E}_0$ is a homotopy equivalence if $X \subset \text{int } N$.

The statements (2) and (3) exhibit a close relation between the restriction map π and the pullback π_* .

2.3. Diffeomorphism groups of 2-manifolds.

Next we recall fundamental facts on diffeomorphism groups of compact 2-manifolds. The following theorem shows that $\mathcal{D}_X(M)_0 \simeq *$ except a few cases. The symbols \mathbb{S}^1 , \mathbb{S}^2 , \mathbb{T} , \mathbb{P} , \mathbb{K} , \mathbb{D} , \mathbb{A} and \mathbb{M} denote the 1-sphere, 2-sphere, torus, projective plane, Klein bottle, disk, annulus and Möbius band respectively.

Theorem 2.3. ([4, 15] etc.) *Suppose M is a compact connected smooth 2-manifold. Then the homotopy type of $\mathcal{D}(M)_0$ is classified as follows:*

M	$\mathcal{D}(M)_0$
\mathbb{S}^2, \mathbb{P}	$SO(3)$
\mathbb{T}	\mathbb{T}
$\mathbb{K}, \mathbb{D}, \mathbb{A}, \mathbb{M}$	\mathbb{S}^1
all other cases	$*$

◦ $\mathcal{D}_\partial(\mathbb{D}) \simeq *$, $\mathcal{D}_\partial(\mathbb{M}) \simeq *$.

◦ If X is a disjoint union of a compact smooth 2-submanifold and finitely many smooth circles and points in M and $\partial M \subset X$, then $\mathcal{D}_X(M)_0 \simeq *$.

For 2-manifolds there is no difference among the conditions: homotopic, C^0 -isotopic, C^∞ -isotopic and joinable by a path in the diffeomorphism group. By [4] and a C^∞ -analogue of [5] we have

Proposition 2.1. *Suppose M is a compact smooth 2-manifold.*

- (1) *Suppose N is a closed collar of ∂M . If $h \in \mathcal{D}_N(M)$ is homotopic to $\text{id}_M \text{ rel } N$, then h is C^∞ -isotopic to $\text{id}_M \text{ rel } N$.*
- (2) *Suppose N is a compact smooth 2-submanifold of M with $\partial M \subset N$. For $h \in \mathcal{D}_N(M)$, the*

following conditions are equivalent:

- (a) h is C^0 -isotopic to id_M rel N .
- (b) h is C^∞ -isotopic to id_M rel N .
- (c) $h \in \mathcal{D}_N(M)_0$.

In Corollaries 2.1 and 2.2 we have a principal bundle with fiber $\mathcal{G} \equiv \mathcal{D}_X(M)_0 \cap \mathcal{D}_N(M)$. The next theorem gives us a sufficient condition that $\mathcal{G} = \mathcal{D}_N(M)_0$. The symbol $\#X$ denotes the cardinal of a set X .

Theorem 2.4. *Suppose M is a compact connected smooth 2-manifold, N is a compact smooth 2-submanifold of M with $\partial M \subset N$, X is a subset of N . Suppose (M, N, X) satisfies the following conditions:*

- (i) $M \neq \mathbb{T}, \mathbb{P}, \mathbb{K}$ or $X \neq \emptyset$.
- (ii) (a) if H is a disk component of N , then $\#(H \cap X) \geq 2$,
(b) if H is an annulus or Möbius band component of N , then $H \cap X \neq \emptyset$,
- (iii) (a) if L is a disk component of $\text{cl}(M \setminus N)$, then $\#(L \cap X) \geq 2$,
(b) if L is a Möbius band component of $\text{cl}(M \setminus N)$, then $L \cap X \neq \emptyset$.

Then we have:

- (1) If $h \in \mathcal{D}_N(M)$ is C^0 -isotopic to id_M rel X , then h is C^∞ -isotopic to id_M rel N .
- (2) $\mathcal{D}(M)_0 \cap \mathcal{D}_N(M) = \mathcal{D}_N(M)_0$.

Theorem 2.4 follows from [17, Theorem 3.1] and Proposition 2.1.

2.4. Basic properties of ANR's and ℓ_2 -manifolds.

The ANR-property of diffeomorphism groups and embedding spaces is also essential in our argument. Here we recall basic properties of ANR's [8, 10, 13] and a topological characterization theorem of ℓ^2 -manifolds.

A metrizable space X is called an ANR (absolute neighborhood retract) for metric spaces if any map $f : B \rightarrow X$ from a closed subset B of a metrizable space Y admits an extension to a neighborhood U of B in Y . If we can always take $U = Y$, then X is called an AR. It is known that X is an AR (an ANR) iff it is a retract of (an open subset of) a normed space. Any ANR has a homotopy type of CW-complex. An AR is exactly a contractible ANR.

We apply the following criterion of ANR's:

Lemma 2.3. (1) *A space X is an ANR iff every point of X has an ANR neighborhood in X .*
(2) *If $X = \bigcup_{i=1}^{\infty} U_i$, U_i is open in X and $U_i \subset U_{i+1}$ and if each U_i is an AR, then X is also an*

- (3) In a fiber bundle, the total space is an ANR iff both the base space and the fiber are ANR's.
 (4) A metric space X is an ANR iff for any $\varepsilon > 0$ there is an ANR Y and maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that gf is ε -homotopic to id_X .

Since any Fréchet space is an AR, every Fréchet manifold is an ANR.

Finally we recall a characterization of ℓ_2 -manifold topological groups [3, 19].

Theorem 2.5. *A topological group is an ℓ_2 -manifold iff it is a separable, non locally compact, completely metrizable ANR.*

The diffeomorphism group $\mathcal{D}(M)_0$ satisfies all conditions except the ANR property (Lemma 2.1). Thus the proof of Theorem 1.1 (1) reduces to the verification of ANR property of $\mathcal{D}(M)_0$. The latter follows from the ANR property of the diffeomorphism groups and embedding spaces of compact 2-manifolds (Lemma 2.2).

3. PROOF OF MAIN THEOREMS

In this section we give a sketch of proofs of Theorems 1.1 and 1.2 in the case where $M \neq$ a plane, an open Möbius band, an open annulus. Below we assume that M is a noncompact connected smooth 2-manifold without boundary and that $M \neq$ a plane, an open Möbius band, an open annulus.

We can write as $M = \bigcup_{i=0}^{\infty} M_i$, where $M_0 = \emptyset$ and for each $i \geq 1$

- (a) M_i is a nonempty compact connected smooth 2-submanifold of M and $M_{i-1} \subset \text{int } M_i$,
- (b) for each component L of $cl(M \setminus M_i)$, L is noncompact and $L \cap M_{i+1}$ is connected.

Note that M is a plane (an open Möbius band, an open annulus) iff infinitely many M_i 's are disks (Möbius bands, annuli respectively). Since $M \neq$ a plane, an open Möbius band, an open annulus, passing to a subsequence, we may assume that

- (c) $M_i \neq$ a disk, an annulus, a Möbius band.

For each $i \geq 1$ let $U_i = \text{int } M_i$, and choose a small closed collar E_i of ∂M_i in $U_{i+1} \setminus U_i$, and set $M'_i = M_i \cup E_i \subset U_{i+1}$.

3.1. Proof of Theorem 1.1.

[1] For each $j > i > k \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc}
(p_{k,j}^i)^*(\mathcal{D}_{M_k \cup (M \setminus U_j)}(M)_0) & \xrightarrow{(p_{k,j}^i)^*} & \mathcal{D}_{M_k \cup (M \setminus U_j)}(M)_0 \\
(\pi_{k,j}^i)^* \downarrow & & \downarrow \pi_{k,j}^i \\
\mathcal{E}_{M_k}(M'_i, U_j)_0 & \xrightarrow{p_{k,j}^i} & \mathcal{E}_{M_k}(M_i, U_j)_0,
\end{array}
\quad \begin{array}{l}
\pi_{k,j}^i, p_{k,j}^i : \text{the restriction maps,} \\
\mathcal{G}_{k,j}^i \equiv \mathcal{D}_{M_k \cup (M \setminus U_j)}(M)_0 \cap \mathcal{D}_{M_i}(M)_0
\end{array}$$

Lemma 3.1. (1) $(\pi_{k,j}^i)^*$ is a principal bundle with fiber $\mathcal{G}_{k,j}^i$.

(2) $\mathcal{G}_{k,j}^i$ is an AR.

(3) $(\pi_{k,j}^i)^*$ is a trivial bundle.

(4) $\mathcal{E}_{M_k}(M'_i, U_j)_0$ is an AR.

In (2) we apply Theorem 2.4 to deduce $\mathcal{G}_{k,j}^i \cong \mathcal{D}_{M_i \cup E_j}(M'_j)_0$. The latter is an AR (Lemma 2.2 (ii), Theorem 2.3).

[2] For each $i > k \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc}
(p_k^i)^*(\mathcal{D}_{M_k}(M)_0) & \xrightarrow{(p_k^i)^*} & \mathcal{D}_{M_k}(M)_0 \\
(\pi_k^i)^* \downarrow & & \downarrow \pi_k^i \\
\mathcal{E}_{M_k}(M'_i, M)_0 & \xrightarrow{p_k^i} & \mathcal{E}_{M_k}(M_i, M)_0,
\end{array}
\quad \begin{array}{l}
\pi_k^i, p_k^i : \text{the restriction maps,} \\
\mathcal{G}_k^i \equiv \mathcal{D}_{M_k}(M)_0 \cap \mathcal{D}_{M_i}(M)_0.
\end{array}$$

Lemma 3.2. (1) $(\pi_k^i)^*$ is a principal bundle with fiber \mathcal{G}_k^i .

(2) $\mathcal{E}_{M_k}(M'_i, M)_0$ is an AR.

(3) $(\pi_k^i)^*$ is a trivial bundle.

(4) $\mathcal{G}_k^i = \mathcal{D}_{M_i}(M)_0$ and $\mathcal{D}_{M_k}(M)_0$ strongly deformation retracts onto $\mathcal{D}_{M_i}(M)_0$.

The assertion (2) follows from Lemma 2.3 (2), Lemma 3.1 (4) and the fact that $\mathcal{E}_{M_k}(M'_i, M)_0 = \cup_{j>i} \mathcal{E}_{M_k}(M'_i, U_j)_0$.

Proof of Theorem 1.1.

(A) $\mathcal{D}(M)_0 \simeq *$:

$\mathcal{D}_{M_i}(M)_0$ strongly deformation retracts onto $\mathcal{D}_{M_{i+1}}(M)_0$ for each $i \geq 0$ (Lemma 3.2 (4)). Since $\text{diam } \mathcal{D}_{M_i}(M)_0 \rightarrow 0$ ($i \rightarrow \infty$), it follows that $\mathcal{D}(M)_0$ strongly deformation retracts onto $\{id_M\}$.

(B) $\mathcal{D}(M)_0$ is an ℓ_2 -manifold:

By Theorem 2.5 and Lemma 2.1 it remains to show that $\mathcal{D}(M)_0$ is an ANR. We apply Lemma 2.3 (4): For each $i \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc}
(p_i)^*(\mathcal{D}(M)_0) & \xrightarrow{(p_i)^*} & \mathcal{D}(M)_0 \\
(\pi_i)^* \downarrow & & \downarrow \pi_i \\
\mathcal{E}(M'_i, M)_0 & \xrightarrow{p_i} & \mathcal{E}(M_i, M)_0,
\end{array}
\quad \begin{array}{l}
\pi_i, p_i : \text{the restriction maps,} \\
q_i : \mathcal{D}(M)_0 \rightarrow (p_i)^*(\mathcal{D}(M)_0) \\
q_i(h) = (h|_{M'_i}, h).
\end{array}$$

Since $(\pi_i)_*$ is a trivial principal bundle with the contractible fiber $\mathcal{D}_{M_i}(M)_0$ (Lemma 3.2 (3),(4) (A)), it follows that $(\pi_i)_*$ admits a section s_i and $s_i(\pi_i)_*$ is $(\pi_i)_*$ -fiber preserving homotopic to id . Consider the two maps

$$\varphi = (\pi_i)_* q_i : \mathcal{D}(M)_0 \rightarrow \mathcal{E}(M_i, M)_0 \quad \text{and} \quad \psi = (p_i)_* s_i : \mathcal{E}(M_i, M)_0 \rightarrow \mathcal{D}(M)_0.$$

Then $\mathcal{E}(M_i, M)_0$ is an ANR (Lemma 2.2 (i)) and $\psi\varphi : \mathcal{D}(M)_0 \rightarrow \mathcal{D}(M)_0$ is π_i -fiber preserving homotopic to id . Since $\text{diam}(\text{fibers of } \pi_i) \rightarrow 0$ ($i \rightarrow \infty$), Lemma 2.3 (4) implies that $\mathcal{D}(M)_0$ is an ANR. \square

3.2. Proof of Theorem 1.2.

We use the following notations:

$$\mathcal{D}_j = \mathcal{D}_{M \setminus U_j}(M)_0, \quad \mathcal{U}_{i,j} = \mathcal{E}(M_i, U_j)_0, \quad \mathcal{U}_{i,j}' = \mathcal{E}(M_i', U_j)_0 \quad (j > i \geq 1).$$

We have the pullback diagram:

$$\begin{array}{ccc} (p_{i,j})^* \mathcal{D}_j & \xrightarrow{(p_{i,j})^*} & \mathcal{D}_j \\ (\pi_{i,j})_* \downarrow & & \downarrow \pi_{i,j} \\ \mathcal{U}_{i,j}' & \xrightarrow{p_{i,j}} & \mathcal{U}_{i,j} \end{array} \quad \begin{array}{l} \pi_i' : \mathcal{D}(M)_0 \rightarrow \mathcal{E}(M_i', M)_0, \\ \pi_{i,j}, p_{i,j}, \pi_i' : \text{the restriction maps.} \end{array}$$

Lemma 3.3. (i) $(\pi_{i,j})_*$ is a trivial bundle with AR fiber.

(ii) $\pi_{i,j}$ has the following lifting property:

(*) If Y is a metric space, B is a closed subset of Y and $\varphi : Y \rightarrow \mathcal{U}_{i,j}'$ and $\varphi_0 : B \rightarrow \mathcal{D}_j$ are map with $p_{i,j}\varphi|_B = \pi_{i,j}\varphi_0$, then there exists a map $\Phi : Y \rightarrow \mathcal{D}_j$ such that $\pi_{i,j}\Phi = p_{i,j}\varphi$ and $\Phi|_B = \varphi_0$.

For each $j > i \geq 1$, we regard as $\mathcal{U}_{i,j}' \subset \mathcal{E}(M_i', M)_0$ and set $\mathcal{V}_{i,j}' = (\pi_i')^{-1}(\mathcal{U}_{i,j}') \subset \mathcal{D}(M)_0$. For each $i \geq 1$ we have:

- (i) $\mathcal{E}(M_i, M)_0 = \bigcup_{j>i} cl \mathcal{U}_{i,j}'$ ($\mathcal{U}_{i,j}'$ is open in $\mathcal{E}(M_i, M)_0$, $cl \mathcal{U}_{i,j}' \subset \mathcal{U}_{i,j+1}'$)
- (ii) $\mathcal{D}(M)_0 = \bigcup_{j>i} cl \mathcal{V}_{i,j}'$ ($\mathcal{V}_{i,j}'$ is open in $\mathcal{D}(M)_0$, $cl \mathcal{V}_{i,j}' \subset \mathcal{V}_{i,j+1}'$, $\mathcal{V}_{i+1,j}' \subset \mathcal{V}_{i,j}'$ ($j > i + 1$))
- (iii) $\mathcal{D}(M)_0^\circ = \bigcup_{j>i} \mathcal{D}_j$ ($\mathcal{D}_j \subset \mathcal{D}_{j+1}$)

Proof of Theorem 1.2.

We construct a homotopy

$$F : \mathcal{D}(M)_0 \times [1, \infty] \rightarrow \mathcal{D}(M)_0 \quad \text{such that} \quad F_\infty = id \quad \text{and} \quad F_t(\mathcal{D}(M)_0) \subset \mathcal{D}(M)_0^\circ \quad (1 \leq t < \infty).$$

(1) F_i ($i \geq 1$): Using Lemma 3.3 (ii), inductively we can construct a map $s_j^i : cl \mathcal{U}_{i,j}' \rightarrow \mathcal{D}_{j+1}$ such that $s_j^i(f)|_{M_i} = f|_{M_i}$ ($f \in cl \mathcal{U}_{i,j}'$) and $s_{j+1}^i|_{cl \mathcal{U}_{i,j}'} = s_j^i$ ($j > i$). Define a map

$s^i : \mathcal{E}(M'_i, M)_0 \rightarrow \mathcal{D}(M)_0^c$ by $s^i|_{cl \mathcal{U}_{i,j}'} = s_j^i$, and set $F_i = s^i \pi'_i$. We have $F_i(cl \mathcal{V}_{i,j}') \subset \mathcal{D}_{j+1}$ and $F_i(h)|_{M_i} = h|_{M_i}$.

(2) F_t ($i \leq t \leq i+1$): Inductively we can construct a sequence of homotopies $G^j : cl \mathcal{V}_{i+1,j}' \times [i, i+1] \rightarrow \mathcal{D}_{j+1}$ ($j > i+1$) such that $G_i^j = F_i$, $G_{i+1}^j = F_{i+1}$, $G^{j+1}|_{cl \mathcal{V}_{i+1,j}' \times [i, i+1]} = G^j$ and $G_i^j(h)|_{M_i} = h|_{M_i}$. If G^j is given, then G^{j+1} is obtained by applying Lemma 3.3(ii) to the diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_0} & \mathcal{D}_{j+2} \\ \cap & & \downarrow \\ Y & \xrightarrow{\varphi} \mathcal{U}_{i,j+2}' \rightarrow \mathcal{U}_{i,j+2}, & \end{array} \quad \varphi(h, t) = h|_{M'_i}, \quad \varphi_0(h, t) = \begin{cases} G_j(h, t) & (h \in cl \mathcal{V}_{i+1,j}') \\ F_i(h) & (t = i, i+1) \end{cases}$$

$$(Y, B) = (cl \mathcal{V}_{i+1,j+1}' \times [i, i+1], (cl \mathcal{V}_{i+1,j}' \times [i, i+1]) \cup (cl \mathcal{V}_{i+1,j+1}' \times \{i, i+1\})).$$

Define $F : \mathcal{D}(M)_0 \times [i, i+1] \rightarrow \mathcal{D}(M)_0^c$ by $F = G^j$ on $cl \mathcal{V}_{i+1,j}' \times [i, i+1]$.

(3) F_∞ : Since $F_t(h)|_{M_i} = h|_{M_i}$ for $t \geq i$, we can continuously extend F by $F_\infty = id$. This completes the proof. \square

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DEPARTMENT OF MATHEMATICS, KYOTO INSTITUTE OF TECHNOLOGY, MATSUGASAKI, SAKYOKU, KYOTO 606, JAPAN

E-mail address: yagasaki@ipc.kit.ac.jp